# Bidirectional Ordered Search

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**ABSTRACT**

We present a bidirectional ordered search model in which consumers sample firms along two opposite paths and the overall search pattern is endogenized by firms' advertising decisions. We demonstrate that advertising level could be a differentiating factor because search order helps filter consumers of different search costs. In equilibrium firms always choose different advertising levels so that consumer traffic is always skewed toward one firm. Asymmetric advertising strategies lead to relaxed price competition and differentiated sales composition. The effect of advertising on profits is moderated by market mobility. We extend the model to consider sequential advertising decision, and show that the equilibrium profits are higher and prices lower.

## 1 Introduction

Classic search literature (e.g., Salop (1977), Varian (1980), Stahl (1989)) usually assumes that search is random and each firm is equally likely to be sampled. In real markets, search is hardly random, but exhibits clear patterns. Firms more advertised, recommended, or prominently-displayed are more likely to be visited before others and they tend to attract higher consumer traffic. Recent years have seen a growing interest in studying restricted search patterns. Arbatskaya (2007) presents an ordered search model in which consumers sample firms in a predetermined order. In her model, sellers located at the entrance of a one-directional market will charge higher prices than those located down the line, because these sellers possess market power over those consumers who will not shop around due to high search costs. Arbatskaya (2007) shows that if sellers bid for positions, in equilibrium the rent from top positions will be completely dissipated. In this article, to further the research on ordered search, we present a bidirectional search model in which consumers sample firms along two opposite directions. In this model, search orders are not exogenously restricted, but arise from firm advertising decisions.

The bidirectional search model shares some common assumptions with that of Arbatskaya (2007), i.e., the product is homogeneous, firms are ex ante identical, and consumers differ only in their search costs. The major difference is that the bidirectional search model considers two firms, each consumer could first evaluate firm 1 or firm 2, which is not predetermined, and the overall consumer search pattern is endogenized by firms' advertising decisions. In the words of the Oriental bazaar analogy in Arbatskaya (2007), both ends of the bazaar could be the entrance and the exit; the probability a consumer chooses one end as the entrance is affected by the firms' advertising levels. In the bidirectional model, the probability a firm is chosen to be first evaluated increases in its own advertising level and decreases in that of its opponent, which captures firm competition for consumer awareness in advertising markets.

The bidirectional search model could be thought to capture the overall traffic pattern that combines the consumer traffic flow in a large number of uni-directional markets. Plenty of research (Arbatskaya, 2007; Athey & Ellison, 2011; Baye & Morgan, 2001; Chen, Iyer, & Padmanabhan, 2002; Ellison & Ellison, 2004; Hagiu & Jullien, 2011; Iyer & Pazgal, 2003; Smith & Brynjolfsson, 2001; Weber & Zheng, 2007; Xu, Chen, & Whinston, 2011, 2012; Zettelmeyer, 2000) has shown that uni-directional markets, such as online referral systems, shopbots, merchant alliances, affiliate programs, pay-for-placement search engines and computer reservation systems, widely exist. In each of these uni-directional markets, consumers evaluate firms according to a given order. However, if firms compete in more than one market, taking all these markets as a whole, search may not be uni-directional. For example, for two competing firms, in one uni-directional market firm 1 is evaluated before firm 2; in another, firm 2 may be evaluated before firm 1. Therefore, the overall search pattern consists of two opposite search paths. Note the bidirectional search model is different from the classic random-search models (e.g., Salop (1977), Varian (1980), Stahl (1989)) in that in the random-search models, each of the firms is equally likely to be sampled, while in the bidirectional model, the probability a firm is first sampled is affected by firm strategy (which is, in this paper, advertising effort). Within this framework, the firm that has a higher advertising level possesses larger market power than the other firm. Our goal is to characterize the equilibrium search pattern and to analyze the impact of advertising competition on prices and firm profits.

We show that in equilibrium consumer search is always asymmetric and one firm draws more traffic than the other. The reason is that one firm always has a higher advertising level and is thus more prominent. That firm also has a higher expected price and more sales. The intuition behind is that high search cost consumers always stop search earlier than lower search cost consumers; thus search order is a filtering mechanism. Choosing differentiated advertising levels and pricing strategies allows firms to target different market segments: the sale of the more advertised firm consists of a larger percentage of higher search cost consumers, while that of the less advertised firm consists of a larger percentage of lower search cost consumers.

Nowadays retailers, especially online ones, spend a considerable amount of budget on generating traffic. Taobao.com, the world's largest electronic marketplace and the second largest Internet advertiser of China, makes almost all money from charging advertising fees from its affiliated sellers, some of which report spending over a third of its revenue on buying consumer traffic. After accounting for advertising costs, do firms have higher or lower profits compared with when firms do not advertise as in the classic random search model? We find that the effect of advertising on profits is moderated by market mobility. Here the market mobility is measured by the percentage of high search cost consumers. When market mobility is high (the percentage of high search cost consumers is low), both firms' profits will be higher than when firms do not advertise because advertising competition is mild and the benefits from differentiation strategies are more than advertising costs. When market mobility is low, firm profits are lower than when firms do not advertise because advertising competition now is intense, advertising levels are high and the benefits from differentiation strategies are less than advertising costs. We also show that lower advertising costs lead to higher prices and higher profits when market mobility is high.

We extend the model to consider firms choosing advertising levels sequentially. We find the follower's decision variable is a strategic complement to that of the leader. In equilibrium both firms set their advertising at a lower level than when the firms choose advertising levels simultaneously, resulting in higher profits for firms and lower prices for consumers. Therefore sequential decision is a cooperating mechanism that benefits all parties.

This article contributes to the growing literature on ordered search (Arbatskaya, 2007; Deck & Wilson, 2006; Perry & Wigderson, 1986). This line of research typically assumes that all consumers follow the same predetermined search order. We present a bidirectional model that does not require such assumption. Our model has a mixed pricing strategy, instead of a pure one as in Arbatskaya (2007). Nevertheless our result is consistent with that of Arbatskaya (2007) in a probabilistic sense: a firm that is more likely to be searched first is more likely to charge a higher price. Our work is different from Arbatskaya (2007) in terms of rent dissipation. Arbatskaya (2007) shows that any benefit from an advantaged position will be completely dissipated in the position auction. In the present paper, firm endogenously decide the amount to spend on acquiring traffic, nevertheless one firm still has a higher profit than the other.

This article also helps us to better understand the role of markets as intermediaries. Previous research (e.g., Baye and Morgan (2001), Chen et al. (2002), Iyer and Pazgal (2003), Hagiu and Jullien (2011)) usually assumes a monopoly market intermediary. By making some firms more prominent than others, the intermediary distorts consumer traffic, relaxes price competition and extracts surplus. This article assumes a competitive advertising market where no one could monopolize advantaged positions. Still, we show that firms voluntarily choose different prominence levels that reflex underlying market mobility, so that consumer traffic is skewed, and price competition relaxed. The amount of surplus extracted by advertising intermediaries depends on market mobility.

The present paper also contributes to the stream of research that combines consumer search and advertising (e.g., Butters (1977), Robert and Stahl (1993)). One implication of the present paper is that a higher advertising level is associated with a higher expected price, whereas in Robert and Stahl (1993), the opposite is true. The reason is advertising serves different function in this paper from the one in Robert and Stahl (1993) . In this paper advertising raises awareness of a firm and directs consumer searches; advertising does not contain price information. In Robert and Stahl (1993) advertising informs consumers about prices; low-price advertising is more effective in attracting consumers.

The rest of the paper is organized as follows. In Section 2, we present the basic assumptions of the model. In section 3, we analyze consumer behavior and firms' pricing and advertising strategies. Section 4 discusses the effects of market mobility and advertising costs on the equilibrium. Section 5 extends the model to sequential decision. Section 6 concludes. Most proofs are collected in the appendix.

## 2 The model assumptions

Consider a market in which consumers search to buy one unit of a homogeneous product. Consumers have the same reservation price $r$. Consumers search firms sequentially, and differ in their search costs. The measure of consumers is normalized to one. A proportion $\gamma$ of consumers, also called high search cost consumers, have positive search cost $c > 0$. The rest $1-\gamma$, also called shoppers or low search cost consumers, have zero search cost. The shoppers are akin to the informed consumers in Varian (1980) or the switchers in Narasimhan (1988). The shopper segment captures the notion that some consumers have a very low opportunity cost of shopping or derive enjoyment from shopping around. The parameter $\gamma$ captures the overall mobility of the market. If $\gamma = 0$, the market is extremely mobile. Price information will be transparent, and all firms will price at the marginal cost. If $\gamma = 1$, all consumers have positive search costs. By Diamond (1971), in equilibrium all firms price at the reservation price and no consumers search. Market mobility is very low.

There are two firms offering the product and facing no capacity constraints. The marginal cost of the product is assumed to be constant and normalized to zero. The firms use advertising to draw consumer awareness and drive consumers to visit their stores. The firms can be interpreted as retailers offering a large assortment of products in a given category, and are mutually substitutable. Firm ads do not contain price information. The firms may employ a large set of media to display their ads at the same time. The media market is highly fragmented and each medium has a miniscule fraction of the whole market. Firms are price takers in advertising markets, so firms are not strategic in pricing ads; the advertising cost function is thus exogenous[1]. Each medium has its own audience and is essentially a small uni-directional market. Audience of the medium browse from prominent slots to inconspicuous slots. The firms need to pay to get their ads placed in the prominent slots, but listed at the inconspicuous slots is free of charge (or normalized to zero). For example, sellers on Taobao.com need to pay for Taobao-Express, an advertising service displaying sellers prominently on navigational pages, but seller showing up in organic search results is free. On a medium, if none of the two firms advertises at prominent slots, each firm has equal likelihood to be searched first by the audience of the medium. If one of the two firms advertises at prominent slots, the audience searches that firm first. If both firms advertise at prominent slots, they impress the audience equally and each firm has equal likelihood to

[1]Readers interested in strategic ad-pricing might refer to Chen and He (2011), Varian (2007), Benjamin Edelman and Schwarz (2007) and Athey and Ellison (2011) for discussions of auction mechanisms in paid search ads.

be searched first. Thus, advertising puts a firm ahead of its opponent in a consumer's search order or neutralizes the position advantage of its opponent. The advertising cost to reach a total $\beta$ coverage of consumers is $A(\beta)$. Based on the constant-reach independent-readership technology (Grossman and Shapiro (1984)), $A(\beta)$ in general should be convexly increasing. Assume $A(0) = 0$, $A'(0) = 0$[2], $A'(\beta) > 0, A''(\beta) > 0$. The cost to reach all consumers, $A'(1)$, is assumed to be very high to ensure the optimal advertising level is always less than 1. Firms choose advertising media independently.

The number of firms, the production and advertising costs and the structure of the game are assumed to be common knowledge. The timing of the game is as follows. First, firms simultaneously choose their advertising levels $\beta_i, i = 1, 2$. Second, after observing the advertising levels, firms simultaneously choose prices $p_i, i = 1, 2$. Third, consumers browse media from prominent slots to inconspicuous slots. Depending on whether a consumer sees a firm's ad at prominent slots, consumers are divided into four groups: (1) a fraction $\beta_1\beta_2$ see the ads of both firms; (2) a fraction $\beta_1(1-\beta_2)$ see the ad of firm 1 but not firm 2; (3) a fraction $\beta_2(1-\beta_1)$ see the ad of firm 2 but not firm 1; (4) a fraction $(1-\beta_1)(1-\beta_2)$ do not see either ads at the prominent slots. This paper considers a rational-expectation equilibrium in which consumers' expectation of firm prices and advertising levels are consistent with firm choices. Group (1) consumers observe both firms' ads, but cannot infer which ad is from the firm that offers lower expected price [3], so group (1) consumers choose to first search either firm with equal likelihood. Group (2) and (3) consumers only observe one firm's ad at prominent slots. Naturally they will follow the ad to search that firm first. Group (4) consumers do not see either ads. They continue to browse inconspicuous slots and have equal likelihood to first search either firm. By this search pattern, the number of consumers that

[2] This can be relaxed at the cost introducing additional corner solutions, but adding no additional insights.

[3] Someone may argue that consumers could check many media to infer firms' advertising levels and then prices, and associate them with ads. Yet we believe that firms' ads may take different forms in different media and this is a large media market. The mental cost of processing such a large volume of information is far more than any saving from such a simple shopping task. Firms, on the other hand, can do so to infer the advertising levels of their opponents, because it is their marketing personnel's regular job to monitor their competitors' moves, and the firms have much at stake.

first search firm $i$ is $\beta_i\beta_j/2 + \beta_i(1-\beta_j) + (1-\beta_i)(1-\beta_j)/2 = (1+\beta_i-\beta_j)/2$. Note that by increasing its advertising level, a firm can increase its traffic and reduce its opponent's traffic, which highlights advertising's redistributive effect on traffic. In particular, if two firms choose the same advertising level, each firm attracts a half consumers, exactly as in random-search models: the effects of advertising are exactly cancelled out. After quoting a price from first search, consumers decide whether to continue searching the next firm. Consumers make purchase decisions after they stop searching. We assume recall is costless and the first price quote is free, which ensures all consumers must search at least once. This technical assumption ensures that the market exists and is commonly used in the literature (e.g., Reinganum (1979), Kuksov (2004)).

# 3 The model analysis

## 3.1 Consumer Behavior

Consider a subgame where consumers make search decisions based on firm pricing strategies. Suppose firm $i$'s pricing strategy is denoted by the cumulative distribution function $F_i(p)$, whose support is $P_i$, upper bound is $u_i$ and lower bound is $l_i$. $F_i(p)$ degenerates into a single price for a pure strategy. $f_i(p)$ is the probability density function associated with $F_i(p)$. In equilibrium $u_i \leq r$ since firms will not charge over consumers' reservation price. In a rational-expectation equilibrium, consumers' expected pricing strategies are consistent with firm choices. However, since ads do not contain price information, consumers cannot associate firm prices with ads before search.

Shoppers have zero search cost. They will search both firms and always purchase from the firm that offers the lower price. High search cost consumers only continue to search when next search yields an expected saving greater than their search costs. We assume they choose to stop search when indifferent between stopping and continuing search. If high search cost consumers have quoted both prices and the prices are equal, they choose to purchase from either firm with equal probability.

Suppose a high search cost consumer first quotes price $z$ from firm $i$. With this information she updates her belief of the identity of firm $i$. By the Bayes' rule,

$$Prob(i = k) = \frac{f_k(z)}{f_1(z) + f_2(z)}, k = 1, 2. \tag{1}$$

Note if a price is only charged by one firm, a consumer that has quoted that price is able to infer the identity of the firm with certainty.

The consumer's expected benefit from searching next firm $j, j \neq i$ is:

$$S(z) = \sum_{j=1,2} Prob(i = 3 - j) \int_{l_j}^{z} (z - x) dF_j(x) = \sum_{j=1,2} \frac{f_{3-j}(z)}{f_1(z) + f_2(z)} \int_{l_j}^{z} F_j(x) dx. \tag{2}$$

The search is worthwhile only if $S(z) > c$. Proposition 1 shows that in equilibrium, for any price $z$, $S(z) \leq c$, so high search cost consumers will not engage next search.

**Proposition 1** *For any price $z$ in the support of the equilibrium pricing strategy, $S(z) \leq c$. Therefore no high search cost consumers will search further after the first price quote.*

To simplify notation, denote the traffic to firm $i$ by $\alpha_i$, $\alpha_i = (1 + \beta_i - \beta_j)/2$. Among $\alpha_i$, $\alpha_i \gamma$ are high search cost consumers and $\alpha_i(1 - \gamma)$ are shoppers.

## 3.2 Pricing Strategies

We now characterize the properties the equilibrium pricing strategies must satisfy.

**Lemma 1** *The lower bounds of $P_1$ and $P_2$ are equal. Define $l \equiv l_1 = l_2$.*

The proof is straightforward: for firm $i$, if $l_i < l_j$, there must exist a price $p$, $l_i \leq p < l_j$, charged with positive density. However, $p$ is strictly dominated by $(p + l_j)/2$. Thus $p$ cannot be charged with positive density. This is a contradiction.

**Lemma 2** *There is no mass at any price* $p < \min\{u_1, u_2\}$. *The support* $P_i$ *is continuous on the interval* $[l, \min\{u_1, u_2\})$, $i = 1, 2$.

Lemma 2 implies that $F_i(p)$ is continuous and strictly increasing for $p < \min\{u_1, u_2\}$. By the lemma, any price $p \in [l, \min\{u_1, u_2\})$ is in the supports of both firms' mixed strategies.

Denote the equilibrium revenue of firm $i$ by $\pi_i$. Given equilibrium strategy $F_j(p)$, the expected revenue of firm $i$ from charging $p \in [l, \min\{u_1, u_2\})$ is :

$$\Pi_i(p) = \pi_i = p(\alpha_i\gamma + (1-\gamma)(1 - F_j(p))). \tag{3}$$

The first component in the parenthesis, $\alpha_i\gamma$, is the number of high search cost consumers diverted to firm $i$. The second component, $(1-\gamma)(1 - F_j(p))$ is the expected number of shoppers, calculated by the number of all shoppers multiplied by the probability firm $j$ charges a price higher than $p$. Firm $j$ does not have a mass at $p$, so the probability that firm $j$ also charges $p$ is zero.

Solving Equation 3 yields firm $j$'s equilibrium strategy:

$$F_j(p) = \frac{1}{1-\gamma}(1 - \gamma + \alpha_i\gamma - \frac{\pi_i}{p}), \quad l \le p < \min\{u_1, u_2\}. \tag{4}$$

By the condition $F_i(l) = 0$, $\pi_i = l(1 - \gamma + \alpha_i\gamma)$. By equation 4, $F_i(p) = \frac{1-\gamma+\alpha_j\gamma}{1-\gamma}(1 - \frac{l}{p})$. Also, since $F_i(p) \le F_j(p)$ when $\alpha_i \ge \alpha_j$, the price of the firm that advertises more first-order stochastically dominates that of the other firm. Therefore, the firm that advertises more charges a higher expected price.

**Lemma 3** *The upper bounds of* $P_1$ *and* $P_2$ *are equal. Define* $u \equiv u_1 = u_2$.

By Lemma 1 and 3, $P_1 = P_2 = [l, u]$. In the proof of Lemma 3, we show only one firm could have a mass at $u$ and the firm that has a mass at $u$ must be the one that advertises more. To simplify notation, in the rest of this subsection we assume $\beta_i \ge \beta_j$ and define $\alpha = \alpha_i$, $0.5 \le \alpha \le 1$. $\alpha$ measures how skewed the consumer traffic is directed to the more advertised firm. $\alpha = 0.5$ implies consumer traffic is completely balanced; $\alpha = 1$ implies all traffic is diverted to one firm.

It remains to specify $l$ and $u$. By equation 3, $\pi_i = u\alpha\gamma = l(\alpha\gamma + 1 - \gamma)$, so $l = u\frac{\alpha\gamma}{1-\gamma+\alpha\gamma}$. Define $u^*$, such that $S_i(u^*) = c$. Since $c = S_i(u^*) = \int_l^{u^*} F_j(x)dx = u^*\Big(1 - \frac{\alpha\gamma}{1-\gamma}\ln\big(\frac{1-\gamma+\alpha\gamma}{\alpha\gamma}\big)\Big)$, $u^* = (1-\gamma)c/(1-\gamma-\alpha\gamma\ln(\frac{\alpha\gamma+1-\gamma}{\alpha\gamma}))$. Therefore $u = \min\{r, u^*\}$. Proposition 2 summarizes the equilibrium pricing strategies.

**Proposition 2** *Given firms' advertising strategies, there is a unique mixed pricing equilibrium,*

$$F_i(p) = \begin{cases} 1 & p \geq u \\ \frac{1-\alpha\gamma}{1-\gamma}(1-\frac{l}{p}) & l \leq p < u \\ 0 & p < l \end{cases}, \tag{5}$$

$$F_j(p) = \begin{cases} 1 & p \geq u \\ \frac{1-\gamma+\alpha\gamma}{1-\gamma}(1-\frac{l}{p}) & l \leq p < u \\ 0 & p < l \end{cases}, \tag{6}$$

*where*

$$l = u\frac{\alpha\gamma}{1-\gamma+\alpha\gamma}, u = \min\{r, u^*\}, \textit{and } u^* = \left\{\frac{(1-\gamma)c}{(1-\gamma-\alpha\gamma\ln(\frac{\alpha\gamma+1-\gamma}{\alpha\gamma}))}\right\}$$

Proposition 2 presents an asymmetric mixed-strategy pricing equilibrium. One unique technical feature of the equilibrium is that its upper bound is capped by the reservation price $r$ and the search cost $c$, while in the previous articles that feature similiar asymmetric mixed-strategy pricing equilibrium (e.g., Narasimhan (1988), Raju, Srinivasan, and R.Lal (1990)), technicality requires $r$ to be the pricing upper bound. Also, when all consumers are the high search cost type (i.e., $\gamma \to 1$), $l \to r, u \to r$, which is consistent with Diamond (1971).

Proposition 2 implies that the upper and the lower bounds of firm prices and the expected prices increase in $\alpha$. It is easy to verify $du^*/d\alpha > 0$, therefore $du/d\alpha \geq 0$ [4]. Similarly $dl/d\alpha > 0$. The expected prices are calculated as follows.

$$Ep_i(\alpha) = \int_l^u pf_i(p)dp = u\frac{\alpha\gamma(1-\alpha\gamma)}{(1-\gamma)(1-\gamma+\alpha\gamma)}\ln\big(\frac{1-\gamma+\alpha\gamma}{\alpha\gamma}\big) + u\frac{(2\alpha-1)\gamma}{1-\gamma+\alpha\gamma}, \qquad (7)$$

$$Ep_j(\alpha) = \int_l^u pf_j(p)dp = u\frac{\alpha\gamma}{1-\gamma}\ln\big(\frac{1-\gamma+\alpha\gamma}{\alpha\gamma}\big). \qquad (8)$$

Hence we have $dEp_i/d\alpha > 0$ and $dEp_j/d\alpha > 0$, which indicates that imbalanced traffic flow relaxes price competition.

**Corollary 1** *More skewed traffic flow leads to higher market prices.*

Asymmetric pricing strategies lead to different mix of consumer types in the final sales for different firms. Suppose the mass firm $i$ charges at $u$ is denoted by $m$, then $m = 1 - F_i^-(u) = (2\alpha-1)\gamma/(1-\gamma+\alpha\gamma)$. Clearly $dm/d\alpha > 0$, and $\lim_{\alpha\to 1} m = \gamma$. The expected sales volume of firm $i$ is

$$Eq_i(\alpha) = \int_l^u q_i(p)f_i(p)dp = \int_l^u \frac{\pi_i}{p}f_i(p)dp = \alpha\gamma + (1-m)\frac{1-\gamma}{2}. \qquad (9)$$

So $dEq_i(\alpha)/d\alpha > 0$. Equation 9 shows that firm $i$ sells to $\alpha\gamma$ high search cost consumers and its expected share of shoppers (calculated by the probability pricing below $u$, $1-m$, times the number of shoppers, $1-\gamma$, and the winning probability 1/2). Since $dm/d\alpha > 0$, greater $\alpha$ implies that firm 1 charges $u$ with higher probability and is more likely to sell to high search cost consumers and less likely to shoppers. The proportion of high search cost consumers in total sales, $\alpha\gamma/Eq_i$, monotonously increases in $\alpha$, from $\gamma$ ($\alpha = 0.5$) to $2\gamma/(1+\gamma^2)$ ($\alpha = 1$). Note $Eq_i < \alpha$; the sales volume is less than the traffic initially directed to firm $i$ because firm $i$ has a net loss of shoppers to firm $j$ due to firm $i$'s higher expected

[4]Strictly speaking, $u$ is not differentiable at the kink point $r = u^*(\alpha)$. The usage in this formula is a slight abuse of notation. Nevertheless we choose to do so to simplify exposition when it does not cause misunderstanding.

price.

The expected sales volume of firm $j$ is

$$Eq_j(\alpha) = \int_l^u q_j(p) f_j(p) dp = \int_l^u \frac{\pi_j}{p} f_j(p) dp = (1-\alpha)\gamma + (1+m)\frac{1-\gamma}{2}. \tag{10}$$

So $dEq_j(\alpha)/d\alpha < 0$. Equation 10 shows that firm $j$ sells to $(1-\alpha)\gamma$ high search cost consumers and its expected share of shoppers, $(1+m)\frac{1-\gamma}{2}$. The proportion of high search cost consumers, $(1-\alpha)\gamma/Eq_j$, monotonously decreases with $\alpha$, from $\gamma$ ($\alpha = 0.5$) to 0 ($\alpha = 1$), which implies that firm $j$ is less likely to sell to high search cost consumers and more likely to shoppers. Note that $Eq_j > 1-\alpha$; the sales volume is greater than the traffic flow initially directed to firm 2 because firm $j$ has a net gain of shoppers from firm $i$ due to firm $j$'s lower expected price.

In summary, we have the following corollary.

**Corollary 2** *More skewed traffic flow leads to higher sales for the more advertised firm and lower sales for the less advertised firm; firm pricing strategies moderate consumer purchase decisions such that buyers of the more/less advertised firm contains a higher/lower percentage of high search cost consumers and a lower/higher percentage of shoppers than the overall consumer composition.*

Our result generalizes Arbatskaya (2007). In Arbatskaya (2007), all consumers follow the same search order and prices decline with the order. We show that, in a bidirectional search market, the result of Arbatskaya (2007) holds in a probabilistic sense: different consumers may have different search orders, but the firm that is placed at the head of more consumers' orders charges a higher expected price than the other firm. Moreover, that firm also has a larger expected sales, which is consistent with Ellison and Ellison (2004) in that ordinal ranking of sellers better predicts sales than prices do.

The effect of $\alpha$ on firm revenue is not straightforward. By Proposition 2, $\pi_i = u\alpha\gamma$, $\pi_j = u\alpha\gamma(1-\alpha\gamma)/(1-\gamma+\alpha\gamma)$. Therefore $\pi_i$ increases in $\alpha$; $\pi_j$, however, is not monotonic in $\alpha$.

Lemma 4 shows that the monotonicity of $\pi_j(\alpha)$ depends on specific parameter configurations.

**Lemma 4** *$\pi_j(\alpha)$ can only be of increasing, unimodal or decreasing shape. In Figure 1, $\pi_j(\alpha)$ of different shapes are mapped into parameter space $\gamma \otimes r$, each shape corresponding to a parameter area labeled with Roman numeral I, II or III. The parameter areas are separated by black solid lines.*

*Area I: when $\gamma \leq (3-\sqrt{5})/2$, or $(3-\sqrt{5})/2 < \gamma \leq \gamma^*$ and $r \geq u^*(1) = (1-\gamma)c/(1-\gamma+\gamma ln(\gamma))$, $\pi_j(\alpha)$ is an increasing function of $\alpha$;*

*Area II: when $(3-\sqrt{5})/2 < \gamma \leq \gamma^*$ and $r < u^*(1)$, or $\gamma^* < \gamma < 2/3$, or $\gamma \geq 2/3$ and $r > u^*(0.5) = (1-\gamma)c/(1-\gamma-0.5\gamma ln(\frac{1-0.5\gamma}{0.5\gamma}))$, $\pi_j(\alpha)$ is a unimodal function of $\alpha$. $\gamma^* \approx 0.529$;*

*Area III: when $\gamma \geq 2/3$ and $r \leq u^*(0.5)$, $\pi_j(\alpha)$ is a decreasing function of $\alpha$.*

Note $u^*(0.5)$ is a function of $\gamma$ with $\alpha = 0.5$. If $r < u^*(0.5)$, the equilibrium price must be bounded by $r$ for any $\gamma$, that is, $u = r$. $u^*(1)$ is a function of $\gamma$ with $\alpha = 1$. If $r > u^*(1)$, the equilibrium price is only bounded by $u^*$ and never by $r$, that is, $u = u^*$. If $u^*(0.5) \leq r \leq u^*(1)$, the equilibrium price is first bounded $u^*$ and then by $r$ when $\alpha$ increases.

Lemma 4 shows that $\pi_j$ actually increases if more consumers first visit firm $i$ ($\alpha$ increases) when the market mobility is high (area I); $\pi_j$ first increases and then decreases in $\alpha$ when the market mobility is intermediate (area II); only when the market mobility is low (area II), $\pi_j$ decreases in $\alpha$.

The effect of $\alpha$ on $\pi_j$ can be explained by two opposite forces summarized in the previous corollaries: relaxed price competition and sales reduction. When the market mobility is high, the relaxed price competition effect outweighs the sales reduction effect; firm $j$ benefits from higher $\alpha$ since the sales loss from the high search cost segment is limited. When the market mobility is low, the sales reduction effect outweighs the relaxed price competition effect; $\pi_j$

decreases in $\alpha$ because the size of the high search cost segment is now significant. When the market mobility is intermediate, there is an optimal traffic flow that balances the two effects and yields the maximal $\pi_j$. In summary the effect of $\alpha$ on $\pi_j$ is moderated by the overall market mobility level.

### 3.3 Advertising Levels

This subsection considers the advertising game. We still use $\alpha$ to denote the traffic to the more advertised firm, that is, $\alpha = (1 + \beta_i - \beta_j)$ if $\beta_i \geq \beta_j$, and $\alpha = (1 + \beta_j - \beta_i)$ if $\beta_i < \beta_j$. Define $\pi_a(\beta_i; \beta_j) = u\alpha\gamma$ and $\pi_b(\beta_i; \beta_j) = u\alpha\gamma(1 - \alpha\gamma)/(1 - \gamma + \alpha\gamma)$, then the payoff of firm $i, i = 1, 2$, can be written as

$$\pi_i^n(\beta_i; \beta_j) = \begin{cases} \pi_a(\beta_i; \beta_j) - A(\beta_i), & \beta_i \geq \beta_j, \\ \pi_b(\beta_i; \beta_j) - A(\beta_i), & \beta_i < \beta_j, \end{cases} \qquad 0 \leq \beta_i \leq 1. \tag{11}$$

**Proposition 3** *The advertising game does not have a symmetric pure strategy equilibrium.*

The intuition behind Proposition 3 is that the payoff function is not quasiconcave at the diagonal line $\beta_i = \beta_j$, so either firm has an incentive to deviate by increasing or decreasing its advertising level.

Next we show this game has asymmetric pure strategy equilibria. Define the best response of firm $i$ to $\beta_j$ as $b_i(\beta_j) = \text{argmax}_{\beta_i} \pi_i^n(\beta_i; \beta_j)$. Lemma 5 characterizes $b_i(\beta_j)$.

**Lemma 5** *$b_i(\beta_j)$ is defined as follows: $b_i(\beta_j) = \text{argmax}_{\beta_i} \pi_a(\beta_i; \beta_j) - A(\beta_i)$ when $\beta_j \leq \hat{\beta}$, and $b_i(\beta_j) = \text{argmax}_{\beta_i} \pi_b(\beta_i; \beta_j) - A(\beta_i)$ when $\beta_j > \hat{\beta}$, where $\hat{\beta}$ is the solution to the equation $\pi_a(b_i(\hat{\beta}); \hat{\beta}) - A(b_i(\hat{\beta})) = \pi_b(b_i(\hat{\beta}); \hat{\beta}) - A(b_i(\hat{\beta}))$. The graph of $b_i(\beta_j)$ is illustrated in Figure 2 and 3.*

*(a) Figure 2 illustrates a typical graph of $b_i(\beta_j)$, where $b_i$ is never the kink point that satisfies $r = u^*$ or the corner solution 0, and $u = u^*$. For $\beta_j \leq \hat{\beta}$, $\beta_j$ is a strategic substitute for $b_i$ ($b_i'(\beta_j) < 0$). For $\beta_j > \hat{\beta}$, $\beta_j$ is a strategic complement to $b_i$ and $0 < b_i'(\beta_j) < 1$ .*

*(b) Figure 3 illustrates two variations of $b_i(\beta_j)$. The chart on the left illustrates the case $b_i$ takes the corner solution* 0 *for some $\beta_j, \beta_j > \hat{\beta}$. The chart on the right illustrates the case $b_i$ is independent of $\beta_j$ ($b_i'(\beta_j) = 0$) when $u = r$; $b_i$ is the kink point that satisfies $r = u^*$ for a small interval of $\beta_j$, where $b_i'(\beta_j) = 1$ .*

The first prominent feature of $b_i(\beta_j)$ is that it is discontinuous: firm $i$ chooses a high advertising level when $\beta_j$ is small, but jumps to a low level when $\beta_j$ reaches $\hat{\beta}$. Technically, if $b_i$ were continuous, there would have been a symmetric pure strategy equilibrium.

Second, when $b_i$ is not the kink point and $b_i > \hat{\beta}$, firm $i$ will plays dovish, not increasing its own advertising level when firm $j$ increases its advertising level. More specifically, $b_i$ is independent of $\beta_j$ when the pricing strategy satisfies $u = r$, because any benefit from increased traffic is exactly cancelled out by the increased advertising cost. When $u = u^*$, firm $i$ actually reduces $b_i$ when $\beta_j$ is increased marginally. If $b_i < \hat{\beta}$, firm $i$ is the less-advertised firm. Firm $i$ will always plays hawkish, increasing $b_i$ as a retaliation to the marginal increase of $\beta_j$, though not to the same extent as the increase of $\beta_j$ ($b_i'(\beta_j) < 1$). The reason firm $i$ changes its reaction is that it now has a much lower marginal advertising cost.

Third, only when $b_i$ happens to be at the kink point, firm $i$ will retaliates with an increase of $b_i$ by the the same extent as the marginal increase of $\beta_j$, which actually does not change traffic pattern, but result in lower profits for both firms.

Fourth, for certain parameters, the less advertised firm chooses not to advertise if the advertising level of the other firm is not too high, even if advertising is free.

By Lemma 5, we prove that $b_i$ and $b_j$ must intersect, and have following proposition.

**Proposition 4** *The advertising game has asymmetric pure strategy equilibria in which the firms always choose different advertising levels.*

Due to the ex-ante symmetry of the firms, the best responses of the firms are symmetric. Proposition 4 shows that the game has either a pair of pure strategy asymmetric equilibria or paired continuum asymmetric equilibria. Without loss of generality, we label the firm with the higher equilibrium advertising level as firm 1 and the other as firm 2 in the rest of the paper.

Figure 4 shows the partition of the parameter space for different equilibria. In area $I$ (the area to the left of the solid connected line), one firm chooses not to advertise ($\beta_2^* = 0$). In area $II$ (the area to the right of the solid connected line), both firms choose positive advertising levels. In the very narrow area between the dotted curves $\bar{\bar{r}}$ and $\bar{r}$, there exists a continuum of equilibria. We summarize these results in Proposition 5.

**Proposition 5** *In equilibrium, one firm chooses not to advertise when the market mobility is sufficiently high even if advertising is free; otherwise both firms advertise. The parameter space that allows for a continuum of asymmetric equilibria is very narrow.*

The asymmetry in advertising implies that consumer traffic is skewed and the market prices are always higher than when no firm advertises. Furthermore, when only one firm advertises, both firms will have higher profits than when no firm advertises. The proof is straightforward. $\pi_1^n(\beta_1^*; 0) > \pi_1^n(0; 0) = \pi_1(0; 0)$. Since $\beta_2^* = 0$ is a corner solution, $\partial_+ \pi_2^n / \partial \beta_2^* |_{\beta_2^*=0} \leq 0$. So $\pi_2^n(0; \beta_1^*) = \pi_2(0; \beta_1^*) > \pi_2(\beta_1^*; \beta_1^*) = \pi_2(0; 0)$.

**Corollary 3** *When only one firm advertises, both firms' profits are higher than when no firm advertises.*

When both firms have positive advertising levels, the equilibrium is a form of Prisoners' dilemma: both could benefit from lowering advertising levels but no one wants to do it unilaterally. If the firms can coordinate by choosing lower advertising levels: $\beta_1 = \beta_1^* - \beta_2^*$ and $\beta_2 = \beta_2^* - \beta_2^* = 0$ respectively, traffic flow will be unchanged, but firm profits will be higher. However, without contractual commitment, such a mutually beneficial agreement would not be realized, since it does not consist of a Nash equilibrium. In general, when the market mobility is so low that the competition for traffic is fierce and advertising levels are high, firms' profits would be lower than when no firm advertises. In a war of attrition, there is no winner.

Another interesting question is whether the more advertised firm makes a higher profit than the other. Corollary 4 shows that as long as $b_i$ never takes the corner solution zero, it is sufficient to prove the more-advertised firm will do better.

**Corollary 4** *The firm with higher advertising level always makes a higher profit than the other firm when not advertising is never a best response.*

# 4 Comparative Statics

## 4.1 Market Mobility

In this subsection we explore the effect of market mobility on the equilibrium. Note that $\gamma$ measures the size of high search cost consumers: greater $\gamma$ implies lower market mobility. Previously we show that there is a continuum of asymmetric equilibria for a narrow parameter range. Assume in that case the firms choose the equilibrium that has the lowest advertising levels and hence highest profits.

**Corollary 5** *$\beta_1^*$ first increases in $\gamma$, then may decrease during a narrow range of $\gamma$, and resumes increasing in $\gamma$, albeit at a slower pace than $\beta_2^*$ does. $\beta_2^*$ is zero when $\gamma$ is below a threshold value, beyond which $\beta_2^*$ always increases in $\gamma$. The traffic flow, $\alpha^*$, is most skewed at an intermediate value of $\gamma$.*

The graphs of $\beta_1^*(\gamma)$, $\beta_2^*(\gamma)$ and $\alpha^*(\gamma)$ are illustrated in Figure 5, where $\theta$ is a parameter associated with the advertising cost function. The graphs of $\pi_1^n(\gamma)$ and $\pi_2^n(\gamma)$ are illustrated in Figure 6, where $u\gamma/2$ is the profit each firm earns when neither advertises. Figure 6 shows the profits of both firms are higher than $u\gamma/2$ when $\gamma$ is relatively small; but when $\gamma$ is sufficiently high, the profits eventually drop below $u\gamma/2$. Note the limiting profit for each firm is $\lim_{\gamma\to 1}\pi_1^n = \lim_{\gamma\to 1}\pi_2^n = r/2 - A(A'^{-1}(r/2))$.

The intuition behind above results is as follows. The less mobile the consumers are, the more valuable they are to firms. However, firms' competition on consumer traffic would dissipate part of the rent from immobile consumers. When the market mobility is high, the competition for consumer traffic is not fierce because different advertising levels relax price competition and differentiate sales composition, which benefit both firms. When the market mobility is low, the differentiation strategy will be less useful and the loss from lost sales will be significant, so consumer traffic competition will be fierce and advertising costs dissipate much of the monopoly rents. When the market mobility is intermediate, firms need to balance differentiation effect and loss sales effect, and choose appropriate advertising levels.

## 4.2 Advertising Costs

The Internet has been constantly innovating new ways to advertise. New advertising media, such as social network website Facebook and social shopping website Meilishuo, have been growing rapidly. These new media offer new opportunities to reach consumers less expensively. We now consider how the change of the advertising cost might affect the equilibrium. Assume the advertising cost is a function of parameter $\theta$. We choose our parameterization such that $\partial A/\partial\theta > 0$ and $\partial^2 A/\partial\beta\partial\theta > 0$, that is, an increases in $\theta$ corresponds to an increase in total and marginal advertising costs. Lower $\theta$ may represent a shift towards to a more efficient advertising technology or increased supply in advertising markets. Assume the simplest form $A(\beta;\theta) = \theta\tilde{A}(\beta)$, where $\theta$ is a multiplicative component that is independent of advertising level. The following corollary summarizes the results.

**Corollary 6** *Lower advertising costs encourage the firms to advertise more except one firm might choose not to advertise. When $\gamma$ is relatively small, $\partial\alpha^*/\partial\theta < 0$. When $\gamma$ is sufficiently large, the sign of $\partial\alpha^*/\partial\theta$ is decided by the sign of $\tilde{A}''(\beta_1^*)/\tilde{A}'(\beta_1^*) - \tilde{A}''(\beta_2^*)/\tilde{A}'(\beta_2^*)$.*

Corollary 6 shows that lower advertising costs will lead to more skewed traffic and milder price competition when the market mobility is high. Firm profits are also higher. This result should not be a surprise because we show previously that when the market mobility is high, differentiation effect dominates sales loss effect. When the market mobility level is low, although lower advertising costs still encourage both firms to advertise more, but its effect on each advertising level depends on the the relative convexity of the advertising function. We can verify that if $\tilde{A}(\beta)$ is a polynomial function (i.e., $\theta\beta^t$), $\tilde{A}''(\beta)/\tilde{A}'(\beta)$ is a decreasing function, so $\partial\alpha^*/\partial\theta < 0$, which implies more skewed traffic and milder competition; if $\tilde{A}(\beta)$ is a logarithmic function (i.e., $-\theta log(1-\beta)$), $\tilde{A}''(\beta)/\tilde{A}'(\beta)$ is an increasing function, so $\partial\alpha^*/\partial\theta > 0$, which implies less skewed traffic and more intense competition.

Grossman and Shapiro (1984) suggest that lower advertising costs encourage firms to advertise more, cause prices to fall and reduce profits. Our finding agrees with Grossman and Shapiro (1984) in that decreased advertising costs lead to higher advertising levels, but we show that the effect on equilibrium price and profit is not clear-cut, but depends on market mobility and/or the convexity of the advertising cost function.

# 5 Sequential Decision

In this section we extend the model to consider firms set adverting levels sequentially. This assumption is sensible since we often observe an industry has recognized leaders which often move first, and advertising contracts usually last for an extended period, and cannot be changed readily, unlike prices.

Suppose firm 1 (the leader) first commits to $\beta_1$; then firm 2 (the follower) observes $\beta_1$ and sets $\beta_2$. Both firms then choose prices simultaneously and consumers behave as described earlier. We solve this game to find the subgame perfect Nash equilibrium. Our purpose is to compare this equilibrium with the simultaneous one. To make the comparison relevant, we focus on the market of relatively low mobility, where not advertising is never the best response of the follower.

We first show that the leader must choose to be the more-advertised firm. Assume the best response of firm 2 is $b_2(\beta_1)$. No matter firm 2 is the more-advertised or less-advertised firm, if $b_2$ is an interior solution to the profit maximization problem, we have $\partial\pi_2^n/\partial\beta_1 = -A'(b_2(\beta_1))$

by the envelope theorem. If $b_2$ is a kink point, we still have $\partial\pi_2^n/\partial\beta_1 = -A'(b_2(\beta_1))$. As before, it can be shown that $0 < b_2'(\beta_1) \leq 1$ if firm 2 is the less-advertised firm and $b_2'(\beta_1) < 0$ is firm 2 is the more-advertised firm. The first order condition of the leader is $d\pi_1^n/d\beta_1 = \partial\pi_1/\partial\beta_1 + \partial\pi_1/\partial b_2 \partial b_2/\partial\beta_1 - A'(\beta_1) = 0$. So $\partial\pi_1/\partial b_2 = A'(\beta_1)/(b_2'(\beta_1) - 1) \leq 0$. Therefore the equilibrium profit of firm 1 never increases in the advertising level of the follower, and it is optimal for the leader to be the more-advertised firm.

Denote the equilibrium advertising levels in the sequential game by $\beta_{1s}^*$ and $\beta_{2s}^*$, and those in the simultaneous game by $\beta_{1c}^*$ and $\beta_{2c}^*$. The first order conditions of firm 1 are $d\pi_1^n/d\beta_{1s}^* = \partial\pi_1/\partial\beta_{1s}^*(1 - b_2'(\beta_{1s}^*)) - A'(\beta_{1s}^*)$=0 and $\partial\pi_1^n/\beta_{1c}^* = \partial\pi_1/\partial\beta_{1c}^* - A'(\beta_{1c}^*) = 0$. Therefore $\beta_{1s}^* < \beta_{1c}^*$. Since here $b_2'(\beta_1) > 0$, we also have $\beta_{2s}^* < \beta_{2c}^*$. Previously we show that either firm's profit decreases in another firm's advertising level, so both firms' profits are higher in the sequential game than in the simultaneous game.

**Proposition 6** *The sequential game has a unique asymmetric pure strategy equilibrium where the leader has a higher advertising level than the follower. Both firms choose lower advertising levels and have higher profits than in the simultaneous game.*

The traffic flow is also less skewed than in the simultaneous game since $b_2'(\beta_1) \leq 1$, so consumers benefit as well because market prices are now lower.

Our result runs contrary to the classic Stackelberg game. In the classic Stackelberg game, quantity is the decision variable. In equilibrium the leader has a higher quantity and a higher profit than in the Cournot game; the follower has a smaller quantity and a lower profit. The key insight that drives the difference is that in our model, the follower's decision variable is a strategic complement to that of the leader; while in the classic Stackelberg game, it is a strategic substitute. In our model the leader foresees that the follower will cooperate by reducing its own advertising level if the leader limits its own advertising. Essentially sequential decision provides a cooperating mechanism that enables the firms to collectively reduce advertising and increase profits.

# 6 Concluding remarks

This paper presents a bidirectional search model in which consumers sample firms along two opposite paths and the overall search pattern is endogenized by firms' advertising decisions. We demonstrate that advertising has both a price effect and a sales-volume effect; in equilibrium firms always choose different advertising levels so that consumer traffic is always skewed toward one firm. In this model advertising just redistributes consumers, not expanding market reach. Traditionally one would expect competition for consumer traffic volume might dissipate some or all of the industry profits. We show that market mobility plays an important role moderating firm decisions and affecting market competitiveness; all firms have even higher profits after accounting for advertising costs when market mobility is high.

This paper demonstrates that advertising level could be a strategic differentiating factor because search order helps filter consumers of different search costs. This insight should carry over to cases where there are many firms and search paths. Symmetric advertising levels and search patterns in general should not consist an equilibrium for each firm has an incentive to differentiate by choosing different prominence levels. Our model may help explain why the markets where some firms advertise much more and much better known than others (e.g. national brands vs. local brands) widely exist and could be sustained in the long term.

Our results have important implications for retailers. Retailers should strategically consider both price and volume effect of their advertising, not blindly mimicking their competitors to attract consumers and engaging unnecessary price wars. When making decisions, firms should be aware of consumer mobility, and if possible, carefully choose the markets to compete in. Lastly, if the industry has a clear leader, moving after the leader actually will benefit the whole industry.

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# Appendix A: Proof of Proposition 1

We first prove that in equilibrium the high search cost consumers who happen to draw the highest prices will not search again, i.e., $S(u_i) \leq c, i = 1, 2$.

If $S(u_i) > c$, the consumers that have quoted $u_i$ will definitely continue to search. Firm $i$ only charges $u_i$ when firm $j$ prices no lower than $u_i$ with positive possibility. Therefore $u_j \geq u_i$.

Assume $u_j = u_i$ and firm $j$ has a mass $w > 0$ at $u_j$. The consumers that have quoted $u_i$ from firm $i$ only has $w/2$ probability buying from firm $i$. In comparison, the consumers that have quoted price $u_i - \epsilon$, where $\epsilon$ is an arbitrarily small value, has $w$ probability buying from firm $i$. Price $u_i$ is strictly dominated by $u_i - \epsilon$, which means $u_i$ can't be charged with positive probability. The consumers that have quoted $u_i$ from firm $j$ will definitely continue to search and all will purchase from from $i$. Therefore it's unwise for firm $j$ to charge $u_i$, which contradicts with firm $j$ having a mass at $u_i$. If firm $j$ doesn't have a mass at $u_j$, the consumers that have quoted $u_i$ from firm $i$ will search and have probability 1 buy from another firm. So it is not optimal for firm $i$ to charge $u_i$ and $u_i$ can not be the upper bound.

Assume $u_j > u_i$. If $S(u_j) > c$, the consumers that have quoted $u_j$ will definitely search again and will be certain to find a lower price from firm $i$ and never return to firm $j$. So we must have $S(u_j) \leq c$. Then charging any price $p_j \in (u_i, u_j)$ is worse than charging price $u_j$. It would be optimal for firm $j$ to charge prices in the interval $(u_i, u_j)$ with zero probability.

However, it also means that firm $i$ can charge a price higher than $u_i$ but smaller than $u_j$ without losing more sales than charging $u_i$, which contradicts with $u_i$ being the upper bound.

In summary, we must have $S(u_i) \le c$.

Now we show that for any price $z < u_i$, we also have $S(z) \le c$. Note that $\int_{l_i}^{z} F_i(x)dx$ increases in $z$. Suppose $S(z) > c$. There must be a firm $i$ whose pricing strategy satisfies $f_i(z)/(f_1(z)+f_2(z)) > f_i(u_i)/(f_1(u_i)+f_2(u_i))$. However, by reducing the density $f_i(z)$, so that $f_i(z)/(f_1(z)+f_2(z)) = f_i(u_i)/(f_1(u_i)+f_2(u_i))$, firm $i$ can always achieve $S(z) < c$ and higher expected profit (because the high search cost consumers that have quoted $z$ will not search), which suggests original strategy is not optimal. Therefore, we must have $S(z) \le c$.

# Appendix B: Proof of Lemma 2

The proof of the lemma is similar to that in Narasimhan (1988) and Jing and Wen (2008). To limit the length of the paper, the proof is omitted.

# Appendix C: Proof of Lemma 3

Assume $u_i > u_j$. On interval $(u_j, u_i)$, firm $i$'s expected payoff is $\Pi_i(p) = p\alpha_i\gamma$, which is an increasing function. Therefore, $(u_j, u_i)$ cannot be in support $P_i$. Since $u_i$ is the upper bound, there must be a mass point at $u_i$. Also, we must have $S(u_i) = c$, otherwise firm $i$ can slightly increase the upper bound, earning a higher payoff.

Firm $i$ cannot have a mass at $u_j$. Assume firm $i$ has a mass $w$ at $u_j$, then firm $j$ can not have a mass at $u_j$ at the same time, since price $u_j - \epsilon$ ($\epsilon$ is an arbitrarily small positive number) yields a higher expected payoff than $u_j$ does. If firm $j$ has no mass at $u_j$, firm $i$ can move the mass $w$ to $u_i$, earning a higher expected payoff, which violates the equilibrium condition.

Assume $\alpha_i \ge \alpha_j$, then $F_i(p) \le F_j(p)$. $S(u_j) \le \int_{l}^{u_j} F_j(x)dx < S(u_i) = c$. Firm $j$ can earn a higher payoff by charging price $u_j + \epsilon$, which violates the condition $u_j$ is the upper bound.

Assume $\alpha_i < \alpha_j$, then $F_i(p) > F_j(p)$. If $S(u_j) < c$, since $(u_j, u_i)$ is not in $P_i$, by charging any price $p$, $u_j < p < u_i$, firm $j$ can earn a higher payoff, which violates the equilibrium condition. Therefore, we must have $S(u_j) = c$. So

$$\begin{aligned} c &= S(u_j) < \int_l^{u_j} F_i(x)dx = \tfrac{1-\gamma+\alpha_j\gamma}{1-\gamma} \int_l^{u_j} \left(1 - \tfrac{l}{x}\right)dx \\ &< \tfrac{1-\gamma+\alpha_j\gamma}{1-\gamma}(1 - \tfrac{l}{u_j})(u_j - l) = \tfrac{1-\gamma+\alpha_j\gamma}{1-\gamma}\left(\tfrac{l}{u_j} - 1\right)^2 u_j \end{aligned}$$

By previous analysis, $S(u_i) = c$, we have

$$c = S(u_i) = u_i - u_j + \int_l^{u_j} F_j(x)dx = u_i - u_j + \frac{1-\gamma+\alpha_i\gamma}{1-\gamma+\alpha_j\gamma} \int_l^{u_j} F_i(x)dx > u_i - u_j + \frac{1-\gamma+\alpha_i\gamma}{1-\gamma+\alpha_j\gamma} c$$

By $\pi_i = l(1-\gamma+\alpha_i\gamma) = u_i\alpha_i\gamma$, $u_i = (1-\gamma+\alpha_i\gamma)l/(\alpha_i\gamma)$. Combining above formulas, we have

$$\Big(\frac{1-\gamma+\alpha_i\gamma}{\alpha_i\gamma} l - u_j\Big) \frac{1-\gamma+\alpha_j\gamma}{(\alpha_j - \alpha_i)\gamma} < c < \frac{1-\gamma+\alpha_j\gamma}{1-\gamma}\Big(\frac{l}{u_j} - 1\Big)^2 u_j$$

Note $\alpha_j = 1 - \alpha_i$. Let $x = l/u_j$, reorganize the above formula, we have

$$\Phi(x) \equiv \frac{(2\alpha_j - 1)\gamma}{1-\gamma}(x-1)^2 - \frac{1-\alpha_j\gamma}{(1-\alpha_j)\gamma} x + 1 > 0.$$

$\Phi(\cdot)$ is a quadratic function with $\Phi(1) = -\frac{1-\alpha_j\gamma}{(1-\alpha_j)\gamma} + 1 = \frac{\gamma-1}{(1-\alpha_j)\gamma} < 0$.

By $\pi_j = l(1-\gamma+\alpha_j\gamma) = u_j(\alpha_j\gamma + (1-\gamma)(1-F_i(u_j)))$, we have $x = \frac{l}{u_j} = \frac{\alpha_j\gamma+(1-\gamma)(1-F_i(u_j))}{1-\gamma+\alpha_j\gamma}$. So, $\frac{\alpha_j\gamma}{1-\gamma+\alpha_j\gamma} < x < 1$. It is easy to verify that $\Phi(\frac{\alpha_j\gamma}{1-\gamma+\alpha_j\gamma}) < 0$. Together with $\Phi(1) < 0$, $\frac{(2\alpha_j-1)\gamma}{1-\gamma} \geq 0$, we conclude that $\Phi(x) < 0$ for $\frac{\alpha_j\gamma}{1-\gamma+\alpha_j\gamma} < x < 1$. However, it contradicts with previous result $\Phi(x) > 0$.

In summary, assuming $u_i > u_j$ only leads to contradictions. Since $i$ could be either firm, we must have $u_1 = u_2$.

# Appendix D: Proof of Lemma 4

When $r \leq u^*(0.5)$, the upper bound is restricted by $r$ for all possible $\alpha$. So, $\pi_j(\alpha) = r\frac{\alpha\gamma(1-\alpha\gamma)}{1-\gamma+\alpha\gamma}$. It can be shown that $\pi_j(\alpha)$ is monotonously increasing in $\alpha$ when $\gamma \leq \frac{3-\sqrt{5}}{2}$; unimodal in $\alpha$ when $\frac{3-\sqrt{5}}{2} < \gamma < \frac{2}{3}$; monotonously decreasing in $\alpha$ when $\frac{2}{3} \leq \gamma < 1$.

When $r \geq u^*(1)$, the upper bound is not restricted by $r$ for all possible $\alpha$. So, $\pi_j(\alpha) = u^*\frac{\alpha\gamma(1-\alpha\gamma)}{1-\gamma+\alpha\gamma}$. By numerical method, it can be shown that $\pi_j(\alpha)$ is monotonously increasing in $\alpha$ when $\gamma \leq \gamma^* \approx 0.53$; unimodal in $\alpha$ when $\gamma^* < \gamma < 1$.

When $u^*(0.5) < r < u^*(1)$, by the monotonicity of $u^*(\alpha, \gamma)$ on $\alpha$, there exists a unique $\hat{\alpha}$ such that $u^*(\hat{\alpha}, \gamma) = r$. Then $u(\alpha, \gamma) = u^*(\alpha, \gamma)$ if $\alpha < \hat{\alpha}$ and $u(\alpha, \gamma) = r$ if $\alpha \geq \hat{\alpha}$.

If $\gamma \leq \frac{3-\sqrt{5}}{2}$, according to previous discussion, no matter $u(\alpha, \gamma) = u^*(\alpha, \gamma)$ or $u(\alpha, \gamma) = r$, $\pi_j(\alpha)$ is monotonously increasing in $\alpha$.

If $\frac{3-\sqrt{5}}{2} < \gamma \leq \gamma^*$, we have $u(\alpha, \gamma) = u^*(\alpha, \gamma)$ and $\pi_j(\alpha)$ increasing in $\alpha$ when $\alpha < \hat{\alpha}$; $u(\alpha, \gamma) = r$ and $\pi_j(\alpha)$ unimodal in $\alpha$ when $\alpha \geq \hat{\alpha}$. Connecting these two sections, we have $\pi_j(\alpha)$ is unimodal in $\alpha$ over $0.5 \leq \alpha \leq 1$.

If $\gamma > \gamma^*$, we have $\pi_j(\alpha) = u^*(\alpha, \gamma)\frac{\alpha\gamma(1-\alpha\gamma)}{1-\gamma+\alpha\gamma}$ when $\alpha < \hat{\alpha}$; $\pi_j(\alpha) = r\frac{\alpha\gamma(1-\alpha\gamma)}{1-\gamma+\alpha\gamma}$ when $\alpha \geq \hat{\alpha}$. In both cases, $\pi_j(\alpha)$ is a unimodal function of $\alpha$. Next, we rule out the possibility that $\pi_j(\hat{\alpha})$ is on the downsloping part of function $u^*(\alpha, \gamma)\frac{\alpha\gamma(1-\alpha\gamma)}{1-\gamma+\alpha\gamma}$ and on the upsloping part of function $r\frac{\alpha\gamma(1-\alpha\gamma)}{1-\gamma+\alpha\gamma}$. The reasoning is simple. If that is so, since $\pi_j(\hat{\alpha})$ is on the downsloping part of function $u^*(\alpha, \gamma)\frac{\alpha\gamma(1-\alpha\gamma)}{1-\gamma+\alpha\gamma}$ and $u^*(\alpha, \gamma)$ is an increasing function of $\alpha$, we must have $\frac{\partial\frac{\alpha\gamma(1-\alpha\gamma)}{1-\gamma+\alpha\gamma}}{\partial\alpha} < 0$, which means $\frac{\partial r\frac{\alpha\gamma(1-\alpha\gamma)}{1-\gamma+\alpha\gamma}}{\partial\alpha} |_{\alpha=\hat{\alpha}} < 0$. That contradicts $\pi_j(\hat{\alpha})$ being on the upsloping part of function $r\frac{\alpha\gamma(1-\alpha\gamma)}{1-\gamma+\alpha\gamma}$. After ruling out this possibility, by combining these two sections, we have $\pi_j$ is unimodal in $\alpha$ over $0.5 \leq \alpha \leq 1$.

# Appendix E: Proof of Proposition 3

Suppose there is an equilibrium $\beta_1^* = \beta_2^* = \beta^*$. $\beta^*$ can't be zero since $A'(0) = 0$ implies each firm would deviate by increasing its advertising intensity. Equilibrium condition requires no firm deviates by increasing or decreasing its advertising intensity. Therefore we must have $\partial_+ \pi_i^n / \partial\beta^* \leq 0$ and $\partial_- \pi_i^n / \partial\beta^* \geq 0$, which suggests that $\partial\pi_b/\partial\beta_{i|\beta_i=\beta_j^*=\beta^*} \geq A(\beta_i^*) \geq \partial\pi_a/\partial\beta_{i|\beta_i=\beta_j^*=\beta^*}$. However, since $-u\alpha\gamma(1-\alpha\gamma)/(1-\gamma+\alpha\gamma) = u\alpha\gamma - u\alpha\gamma(2-\gamma)/(1-\gamma+\alpha\gamma)$ and hence $-d(u\alpha\gamma(1-\alpha\gamma)/(1-\gamma+\alpha\gamma))/d\alpha < d(u\alpha\gamma)/d\alpha$, we have $\partial\pi_b(\beta_j)/\partial\beta_j < \partial\pi_a(\beta_i)/\partial\beta_i$ and $\partial\pi_b(\beta_i)/\partial\beta_{i|\beta_i=\beta_j^*=\beta^*} = \partial\pi_b(\beta_j)/\partial\beta_{j|\beta_j=\beta_i^*=\beta^*} < \partial\pi_a(\beta_i)/\partial\beta_{i|\beta_i=\beta_j^*=\beta^*}$, which contradicts the previous result. Therefore, there does not exist a symmetric pure strategy equilibrium. In essence the nonexistence is due to the failure to hold quasiconcavity condition along the diagonal of payoffs.

# Appendix F: Proof of Lemma 5

First we restrict the strategy space to $\beta_i \geq \beta_j$. If $u = r$, $\pi_i^n(\beta_i)$ is clearly concave. If $u = u^*$, by $A'(0) = 0, A'(1)$ is arbitrarily large, there must be a local maximizer for $\pi_i^n$; $\pi_i^n(\beta_i)$ is also concave at the maximizer. If for certain $\beta_j$, there is a value $\beta_i^\dagger$ that satisfies $r = u^*$, $\pi_i^n(\beta_i)$ will not be differentiable at $\beta_i^\dagger$; we can show $\partial_- \pi_i^n / d\beta_i^\dagger > \partial_+ \pi_i^n / d\beta_i^\dagger$; therefore $\pi_i^n(\beta_i)$ is also concave at $\beta_i^\dagger$. The best response $b_i(\beta_j)$ could be either an interior solution or a corner solution (kink point) that satisfies $r = u^*$. If $b_i(\beta_j)$ is an interior maximizer to the profit function with $u = r$, the first order condition will show that $b_i$ is independent of $(\beta_j)$; in other words, $b_i'(\beta_j) = 0$ and its graph is horizontal. If $b_i(\beta_j)$ is an interior maximizer to the profit function with $u = u^*$, by the first order condition and the implicit function theorem, $b_i'(\beta_j) < 0$ and its graph is downward-sloping. If $b_i(\beta_j)$ is the kink-point, $b_i(\beta_j) = 2u^{*-1}(r) + \beta_j - 1$, therefore $b_i'(\beta_j) = 1$. For all above cases, we show $\partial\pi_i^n(b_i(\beta_j))/\partial\beta_j = -A'(b_i(\beta_j)) < 0$ by the envelope theorem. We represent the best response function derived under condition $\beta_i \geq \beta_j$ by $b_i^{(1)}(\beta_j)$.

Now consider the strategy space $\beta_i < \beta_j$. For either $u = r$ or $u = u^*$, $\pi_i^n(\beta_i)$ is a unimodal function of $\beta_i$. At the kink point $\beta_i^\dagger$ that satisfies $r = u^*((1+\beta_j-\beta_i^\dagger)/2)$, we show $\partial_- \pi_i^n / \partial\beta_i^\dagger > \partial_+ \pi_i^n / \partial\beta_i^\dagger$. So $\pi_i^n$ must be a unimodal function of $\beta_i$ in general. If the best response $b_i(\beta_j)$ is an interior maximizer to the profit function with either $u = r$ or $u = u^*$, by the first order condition and the implicit function theorem, $0 < b_i'(\beta_j) < 1$, so its graph is upward-sloping. If $b_i(\beta_j)$ is the kink-point, $b_i(\beta_j) = 1 + \beta_j - 2u^{*-1}(r)$, $b_i'(\beta_j) = 1$. For above cases, $\partial\pi_i^n(b_i(\beta_j))/\partial\beta_j = -A'(b_i(\beta_j)) < 0$ by the envelope theorem. $b_i(\beta_j)$ has a corner solution 0 when $\pi_i^{n'}(\beta_i = 0) < 0$. Then $\partial\pi_i^n(b_i(\beta_j))/\partial\beta_j > 0$. We represent the best response function derived under condition $\beta_i < \beta_j$ by $b_i^{(2)}(\beta_j)$.

According to above results, $b_i^{(1)}(\beta_j)$ is always non-increasing except for an possible interval, so it must intersect the diagonal $\beta_i = \beta_j$ at certain value $\bar{\beta}$, which implies $b_i^{(1)}(\bar{\beta}) = \bar{\beta}$. $b_i^{(2)}(\beta_j)$ never intersects $\beta_i = \beta_j$ for $\beta_j > 0$ since $b_i^{'(2)}(\beta_j) < 1$ in general and $b_i^{'(2)}(\beta_j) = 1$ just for an possible interval. Therefore $\pi_i^n(b_i^{(2)}(\bar{\beta}); \bar{\beta}) > \pi_i^n(\bar{\beta}; \bar{\beta})$. Since $\pi_a(\bar{\beta}; \bar{\beta}) = \pi_b(\bar{\beta}; \bar{\beta})$, we have $\pi_i^n(b_i^{(2)}(\bar{\beta}); \bar{\beta}) > \pi_i^n(b_i^{(1)}(\bar{\beta}); \bar{\beta})$. Also, it is easy to see that $\pi_i^n(b_i^{(1)}(0); 0) > \pi_i^n(0; 0) = \pi_i^n(b_i^{(2)}(0); 0)$. By continuity, $\pi_i^n(b_i^{(1)}(\beta))$ and $\pi_i^n(b_i^{(2)}(\beta))$ must cross at a point $\hat{\beta}$, such that $\pi_i^n(b_i^{(1)}(\hat{\beta}); \hat{\beta}) = \pi_i^n(b_i^{(2)}(\hat{\beta}); \hat{\beta})$. What's more, by previous analysis, $\partial\pi_i^n(b_i^{(1)}(\beta_j))/\partial\beta_j < \partial\pi_i^n(b_i^{(2)}(\beta_j))/\partial\beta_j$. So $\hat{\beta}$ must be unique. Therefore, the general best response function is $b_i = b_i^{(1)}$ when $\beta_j \leq \hat{\beta}$ and $b_i = b_i^{(2)}$ when $\beta_j > \hat{\beta}$.

# Appendix G: Proof of Proposition 4

In another word, we need to prove that $b_i$ and $b_j$ must intersect. To do that, it is sufficient to prove that, suppose $b_i^{(1)}$ intersect $b_j^{(2)}$ at $(\beta_i^*, \beta_j^*)$, we must have $\pi_i^n(\beta_i^*; \beta_j^*) > \pi_i^n(b_i^{(2)}(\beta_j^*); \beta_j^*)$. By Lemma 5, $b_i^{(1)}$ and $b_j^{(2)}$ must intersect and $\beta_i^* - \beta_j^* > \beta_j^* - b_i^{(2)}(\beta_j^*)$. Define $x, 0 < x \leq \beta_j^* - b_i^{(2)}(\beta_j^*)$, we have $\partial\pi_i^n(\beta_i; \beta_j^*)/\partial\beta_i|_{\beta_i = \beta_j^* + x} \geq \partial_+\pi_i^n(\beta_i; \beta_j^* + x)/\partial\beta_i|_{\beta_i = \beta_j^* + x}$ since $\partial^2\pi_i^n/\partial\beta_i\partial\beta_j \leq 0$; $\pi_i^{n'}(\beta_j^* - x; \beta_j^*) > \pi_i^{n'}(\beta_j^*; \beta_j^* + x)$ since $A'(\beta_j^* - x) < A'(\beta_j^*)$. By the previous result, we have $\partial_+\pi_i(\beta_i; \beta_j^* + x)/\partial\beta_i|_{\beta_i = \beta_j^* + x} > -\partial_-\pi_i(\beta_i; \beta_j^* + x)/\partial\beta_i|_{\beta_i = \beta_j^* + x}$, so $\partial_+\pi_i^n(\beta_i; \beta_j^* + x)/\partial\beta_i|_{\beta_i = \beta_j^* + x} > -\partial_-\pi_i^n(\beta_i; \beta_j^* + x)/\partial\beta_i|_{\beta_i = \beta_j^* + x}$. Besides, we have $\partial_-\pi_i^n(\beta_i; \beta_j^* + x)/\partial\beta_i|_{\beta_i = \beta_j^* + x} < \pi_i^{n'}(\beta_j^*; \beta_j^* + x)$ since $b_i^{(2)}(\beta_j^* + x) \leq \beta_j^*$. Combining above inequations, we have $\partial\pi_i^n(\beta_i; \beta_j^*)/\partial\beta_i|_{\beta_i = \beta_j^* + x} > -\pi_i^{n'}(\beta_j^* - x; \beta_j^*)$ and $d\pi_i^n(\beta_j^* + x; \beta_j^*)/dx > d\pi_i^n(\beta_j^* -$

$x;\beta_j^*)/dx$. Therefore, $\pi_i^n(\beta_i^*;\beta_j^*) > \pi_i^n(2\beta_j^* - b_i^{(2)}(\beta_j^*);\beta_j^*) = \pi_i^n(\beta_j^*;\beta_j^*) + \int_0^{\beta_j^*-b_i^{(2)}(\beta_j^*)} d\pi_i^n(\beta_j^* + x;\beta_j^*)/dxdx > \pi_i^n(\beta_j^*;\beta_j^*) + \int_0^{\beta_j^*-b_i^{(2)}(\beta_j^*)} d\pi_i^n(\beta_j^* - x;\beta_j^*)/dxdx = \pi_i^n(b_i^{(2)}(\beta_j^*);\beta_j^*)$.

# Appendix H: Proof of Proposition 5

In this proof we characterize the equilibria and the partition of the parameter space. Define $\bar{\beta}_1 = \text{argmax}_{\beta_1}\{r\alpha\gamma - A(\beta_1)\}$, $\bar{\bar{\beta}}_1 = \text{argmax}_{\beta_1}\{u^*\alpha\gamma - A(\beta_1)\}$, $\bar{\beta}_2 = \text{argmax}_{\beta_2}\{r\phi(\alpha) - A(\beta_2)\}$ and $\bar{\bar{\beta}}_2 = \text{argmax}_{\beta_2}\{u^*\phi(\alpha) - A(\beta_2)\}$, where $\phi(\alpha) = \alpha\gamma(1-\alpha\gamma)/(1-\gamma+\alpha\gamma)$; $\beta_1^\dagger$ and $\beta_2^\dagger$ are kink points that satisfy $r = u^*((1+\beta_1^\dagger - \beta_2^\dagger)/2)$.

The equilibrium advertising strategy $(\beta_1^*, \beta_2^*)$ should satisfy both reaction functions, $\beta_1^* = b_1(\beta_2^*), \beta_2^* = b_2(\beta_1^*)$. Consider the kink points $\beta_1^\dagger, \beta_2^\dagger$ that satisfy $\partial_+\pi_1^n/\partial\beta_1^\dagger = 0$ and $\partial_-\pi_2^n/\partial\beta_2^\dagger = 0$; we define $\bar{r} = u^*((1+\beta_1^\dagger - \max\{\beta_2^\dagger, 0\})/2)$. Consider the kink points $\beta_1^\dagger, \beta_2^\dagger$ that satisfy $\partial_-\pi_1^n/\partial\beta_1^\dagger = 0$ and $\partial_+\pi_2^n/\partial\beta_2^\dagger = 0$; we define $\bar{\bar{r}} = u^*((1+\beta_1^\dagger - \max\{\beta_2^\dagger, 0\})/2)$. Then $\beta_1^* = b_1(\beta_2^*) = \bar{\beta}_1$ and $\beta_2^* = b_2(\beta_1^*) = \max\{\bar{\beta}_2, 0\}$ when when $r < \bar{r}$. $\beta_1^* = b_1(\beta_2^*) = \bar{\bar{\beta}}_1^*$ and $\beta_2^* = b_2(\beta_1^*) = \max\{\bar{\bar{\beta}}_2^*, 0\}$ when $r > \bar{\bar{r}}$; $\beta_1^* = b_1(\beta_2^*) = \beta_1^{\dagger *}$ and $\beta_2^* = b_2(\beta_1^*) = \max\{\beta_2^{\dagger *}, 0\}$ with $r = u((1+\beta_1^* - \beta_2^*)/2)$ when $\bar{r} \le r \le \bar{\bar{r}}$. In summary, the equilibrium strategies are

$$(\beta_1^*, \beta_2^*) = \begin{cases} (\bar{\beta}_1^*, \max\{\bar{\beta}_2^*, 0\}), & r < \bar{r} \\ (\beta_1^{\dagger *}, \max\{\beta_2^{\dagger *}, 0\}), & \bar{r} \le r \le \bar{\bar{r}} \\ (\bar{\bar{\beta}}_1^*, \max\{\bar{\bar{\beta}}_2^*, 0\}), & r > \bar{\bar{r}} \end{cases} \tag{12}$$

In particular, for $\beta_1^* = \beta_1^{\dagger *}, \beta_2^* = \beta_2^{\dagger *} > 0$, the sufficient condition is equation $\beta_1^{\dagger *} - \beta_2^{\dagger *} = 2u^{*-1}(r) - 1$, and inequations $\partial_-\pi_1^n/\partial\beta_1^{\dagger *} \ge 0$, $\partial_+\pi_1^n/\partial\beta_1^{\dagger *} \le 0$, $\partial_-\pi_2^n/\partial\beta_2^{\dagger *} \ge 0$, and $\partial_+\pi_2^n/\partial\beta_2^{\dagger *} \le 0$. There exist a continuum of equilibria that satisfy the sufficient condition. Among the equilibria, the one with the smallest advertising levels are preferred by both firms for it yields the highest profits for both firms.

The separating solid connected line is derived as follows. The condition for $\beta_2^* = 0$ is $\partial_+\pi_2^n/\partial\beta_2|_{\beta_2=0,\beta_1=b_1(0)} \le 0$. When $r \le \bar{\bar{r}}$, $u = r$; the graph of the boundary condition

$\partial_+\pi_2^n/\partial\beta_2^*|_{\beta_2^*=0,\beta_1^*=b_1(0)} = 0$ is curve ①. When $r > \bar{\bar{r}}$, $u = u^*$; the graph of the boundary condition $\partial_+\pi_2^n/\partial\beta_2^*|_{\beta_2^*=0,\beta_1^*=b_1(0)} = 0$ is line ②. Note here $u = u^*$, so line ② is a straight line.

# Appendix I: Proof of Corollary 4

If $b_i$ is an interior solution to the profit maximization problem, by the envelope theorem, we have $\partial\pi_i^n/\partial\beta_j = -A'(b_i(\beta_j))$. If $b_i$ is a kink point, we still have $\partial\pi_i^n/\partial\beta_j = -A'(b_i(\beta_j))$. Therefore, given $\beta_i$ is the best response to $\beta_j$, $\pi_i^n$ always decreases in $\beta_j$. Hence at equilibrium $(\beta_i^*, \beta_j^*)$ with $\beta_i^* > \beta_j^*$, $\pi_i^n(\beta_i^*; \beta_j^*) > \pi_i^n(\beta_j^*; \beta_i^*) = \pi_j^n(\beta_j^*; \beta_i^*)$.

# Appendix J: Proof of Corollary 5

When $\gamma$ is small so that the parameter is in area I of Figure 4, firm 2, the less advertised firm, will choose the corner solution: not to advertise at all, and the other will choose $\beta_1^* = b_1(0)$. It can be verified that $\partial\beta_1^*/\partial\gamma > 0$. When $\gamma$ is sufficiently large, the parameter will eventually fall into the part $r < \bar{r}$ in area II, where $u = r$. By the system of first order conditions and the implicit function theorem, the derivatives are then given by

$$\begin{pmatrix} \frac{\partial\beta_1^*}{\partial\gamma} \\ \frac{\partial\beta_2^*}{\partial\gamma} \end{pmatrix} = -H^{-1} \begin{pmatrix} \frac{\partial^2\pi_1^n}{\partial\beta_1^*\partial\gamma} \\ \frac{\partial^2\pi_2^n}{\partial\beta_2^*\partial\gamma} \end{pmatrix}, H = \begin{pmatrix} \frac{\partial^2\pi_1^n}{\partial\beta_1^{*2}} & \frac{\partial^2\pi_1^n}{\partial\beta_1^*\partial\beta_2^*} \\ \frac{\partial^2\pi_2^n}{\partial\beta_2^*\partial\beta_1^*} & \frac{\partial^2\pi_2^n}{\partial\beta_2^{*2}} \end{pmatrix} \tag{13}$$

$H$ is the Hessian matrix. By the optimality condition, the equilibirum requires $\partial^2\pi_1^n/\partial\beta_1^{*2} < 0$, $\partial^2\pi_2^n/\partial\beta_2^{*2} < 0$, $|H| > 0$. It can be verified that $\partial^2\pi_1^n/\partial\beta_1\partial\beta_2 = 0$, $\partial^2\pi_1^n/\partial\beta_1\partial\gamma > 0$, and $\partial^2\pi_2^n/\partial\beta_1\partial\beta_2 > 0$, $\partial^2\pi_2^n/\partial\beta_2\partial\gamma > 0$. By Equation 13,

$$\begin{aligned} \frac{\partial\beta_1^*}{\partial\gamma} &= -(\frac{\partial^2\pi_2^n}{\partial\beta_2^{*2}}\frac{\partial^2\pi_1^n}{\partial\beta_1^*\partial\gamma} - \frac{\partial^2\pi_1^n}{\partial\beta_1^*\partial\beta_2^*}\frac{\partial^2\pi_2^n}{\partial\beta_2^*\partial\gamma})/|H| = -\frac{\partial^2\pi_2^n}{\partial\beta_2^{*2}}\frac{\partial^2\pi_1^n}{\partial\beta_1^*\partial\gamma}/|H| > 0, \\ \frac{\partial\beta_2^*}{\partial\gamma} &= -(-\frac{\partial^2\pi_2^n}{\partial\beta_1^*\partial\beta_2^*}\frac{\partial^2\pi_1^n}{\partial\beta_1^*\partial\gamma} + \frac{\partial^2\pi_1^n}{\partial\beta_1^{*2}}\frac{\partial^2\pi_2^n}{\partial\beta_2^*\partial\gamma})/|H| > 0. \end{aligned} \tag{14}$$

Therefore the advertising levels of both firms increase with $\gamma$. On a side note, the limiting values are : $\lim_{\gamma\to 1}\beta_1^* = \lim_{\gamma\to 1}\beta_2^* = A'^{-1}(r/2)$.

The sign of $\partial\beta_i^*/\partial\gamma$ can't be resolved through analytical methods when the parameter is in other part of area II. We verified through extensive numerical simulations with commonly used convex cost functions, such as polynomial, exponential, log-square (such as $\theta ln^2(1-\beta)$), and found that $\partial\beta_2^*/\partial\gamma > 0$, but $\beta_1^*$ is not monotonic: $\partial\beta_1^*/\partial\gamma > 0$ when $r > \bar{\bar{r}}$; $\partial\beta_1^*/\partial\gamma < 0$ when $\bar{r} \le r \le \bar{\bar{r}}$.

Next we analyze the traffic flow $\alpha^*$. When the parameter is in area I of Figure 4, $\partial\beta_1^*/\partial\gamma > 0, \beta_2^* = 0$; so $\partial\alpha^*/\partial\gamma > 0$. When $\gamma$ is sufficiently large, the parameter falls into part $r < \bar{r}$ of area II. Then

$$\frac{\partial\alpha^*}{\partial\gamma} = \frac{\partial\beta_1^* - \partial\beta_2^*}{2\partial\gamma} = (A''(\beta_2^*)\frac{\partial^2\pi_1}{\partial\beta_1^*\partial\gamma} - A''(\beta_1^*)\frac{\partial^2\pi_2}{\partial\beta_2^*\partial\gamma})/2|H| \tag{15}$$

The formula

$$\frac{\partial^2\pi_2}{\partial\beta_2\partial\gamma} = -r\frac{\partial^2\phi(\alpha)}{2\partial\alpha\partial\gamma} = r\frac{-1+\gamma+5\alpha\gamma+3(\alpha-2)\alpha\gamma^2+\alpha(2-3\alpha+\alpha^2)\gamma 3}{2(1-(1-\alpha)\gamma)^3}$$

is greater than $r/2$ for any $\alpha$ when $\gamma > 0.6$. So $\partial^2\pi_2/\partial\beta_2\partial\gamma > \partial^2\pi_1/\partial\beta_1\partial\gamma = r/2$; therefore $\partial\alpha^*/\partial\gamma < 0$ when $\gamma$ is sufficiently large. This result implies that the advertising intensity of firm 2 will grow faster than that of firm 1, and the skewness of the traffic flow will become smaller when $\gamma$ is sufficiently large. The limiting value is $\lim_{\gamma\to 1}\alpha^* = (1 + \lim_{\gamma\to 1}\beta_1^* - \lim_{\gamma\to 1}\beta_2^*)/2 = 0.5$. This analysis shows that the maximum of $\alpha^*$ must be reached at an intermediate $\gamma$. The consumer traffic flow is most skewed at an intermediate proportion of high search cost consumers.

# Appendix K: Proof of Corollary 6

When the parameter is in area I of Figure 4, it is straightforward to show that $\partial\beta_1^*/\partial\theta < 0$. When $\beta_i^*$ is the interior solution, by the system of first order conditions and the implicit function theorem, the derivatives are then given by

$$\begin{pmatrix} \frac{\partial\beta_1^*}{\partial\theta} \\ \frac{\partial\beta_2^*}{\partial\theta} \end{pmatrix} = -H^{-1} \begin{pmatrix} \frac{\partial^2\pi_1^n}{\partial\beta_1^*\partial\theta} \\ \frac{\partial^2\pi_2^n}{\partial\beta_2^*\partial\theta} \end{pmatrix}. \tag{16}$$

When the parameter is in area II and $r < \bar{r}$, since $\partial^2\pi_i^n/\partial\beta_i\partial\theta = -\partial^2\tilde{A}/\partial\beta_i\partial\theta < 0$, we have

$$\begin{aligned} \frac{\partial\beta_1^*}{\partial\theta} &= -(\frac{\partial^2\pi_2^n}{\partial\beta_2^{*2}}\frac{\partial^2\pi_1^n}{\partial\beta_1^*\partial\theta} - \frac{\partial^2\pi_1^n}{\partial\beta_1^*\partial\beta_2^*}\frac{\partial^2\pi_2^n}{\partial\beta_2^*\partial\theta})/|H| < 0, \\ \frac{\partial\beta_2^*}{\partial\theta} &= -(-\frac{\partial^2\pi_2^n}{\partial\beta_1^*\partial\beta_2^*}\frac{\partial^2\pi_1^n}{\partial\beta_1^*\partial\theta} + \frac{\partial^2\pi_1^n}{\partial\beta_1^{*2}}\frac{\partial^2\pi_2^n}{\partial\beta_2^*\partial\theta})/|H| < 0. \end{aligned}$$

When the parameter is in other part of area II. We verified through comprehensive numerical simulations with commonly used convex cost functions, such as polynomial, exponential and logarithmic and found that $\partial\beta_i^*/\partial\theta < 0$. Therefore, decreases in advertising costs always induce firms to increase advertising levels (except when firm 2 chooses not to advertise).

When the parameter is in area I of Figure 4, we have $\partial\alpha^*/\partial\theta < 0$ since $\partial\beta_1^*/\partial\theta < 0$. When $\beta_i^*$ is the interior solution, $\partial\alpha^*/\partial\theta = (\partial\beta_1^*/\partial\theta - \partial\beta_2^*/\partial\theta)/2 = (\tilde{A}''(\beta_1^*)\tilde{A}'(\beta_2^*) - \tilde{A}''(\beta_2^*)\tilde{A}'(\beta_1^*))/(2|H|)$. Therefore the sign of $\partial\alpha^*/\partial\theta < 0$ is decided by the sign of $\tilde{A}''(\beta_1^*)/\tilde{A}'(\beta_1^*) - \tilde{A}''(\beta_2^*)/\tilde{A}'(\beta_2^*)$.

**Figure 1:** Partition of the monotonicity of $\pi_2(\alpha)$

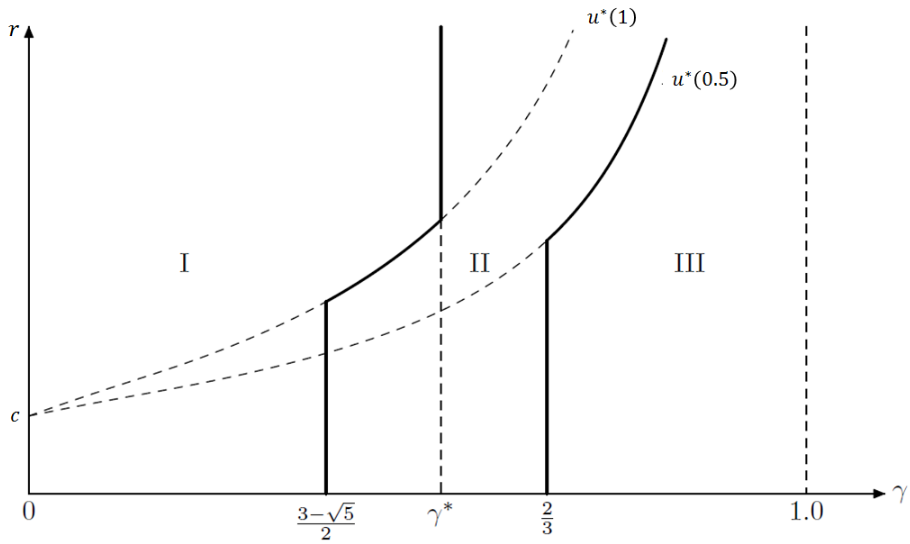


**Figure 2:** General form of the best response function

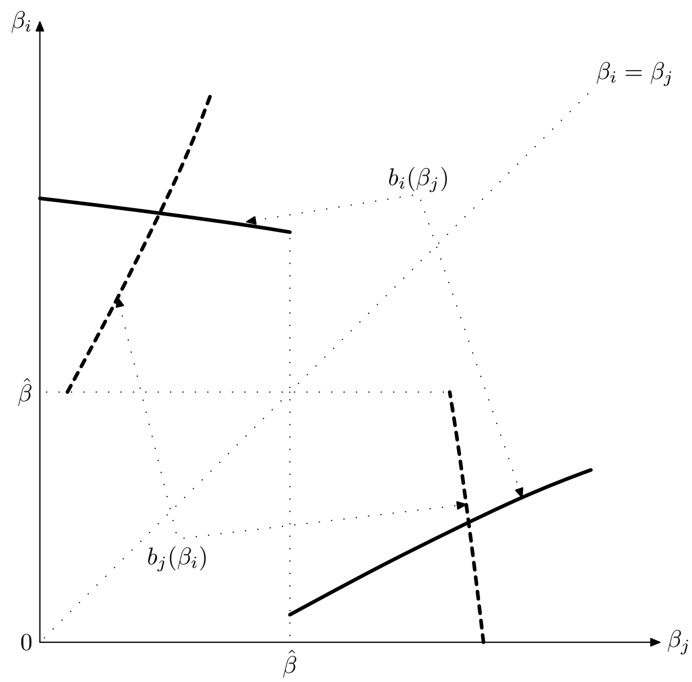

**Figure 3:** Variations of the best response function

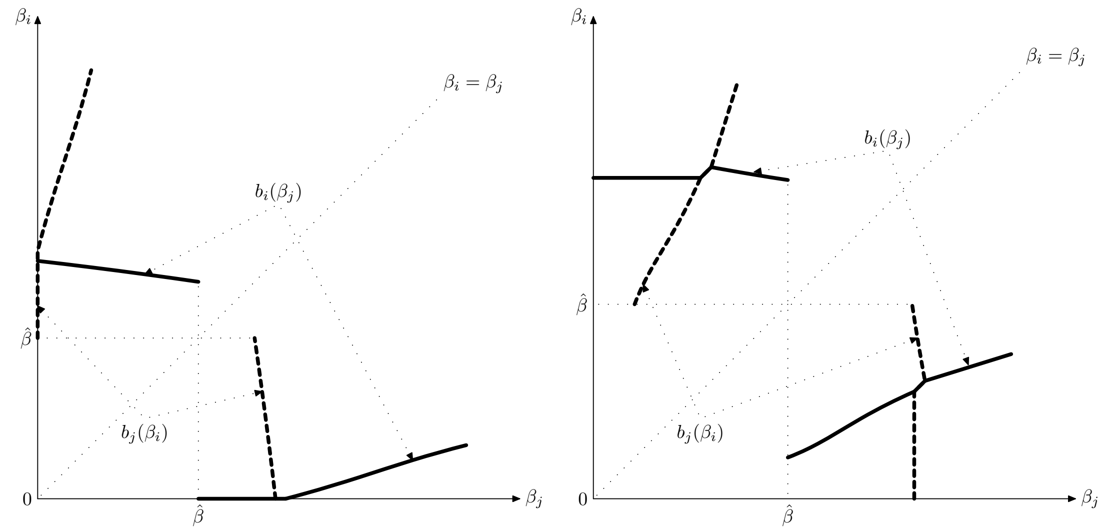


**Figure 4:** Partition of the equilibrium advertising strategies

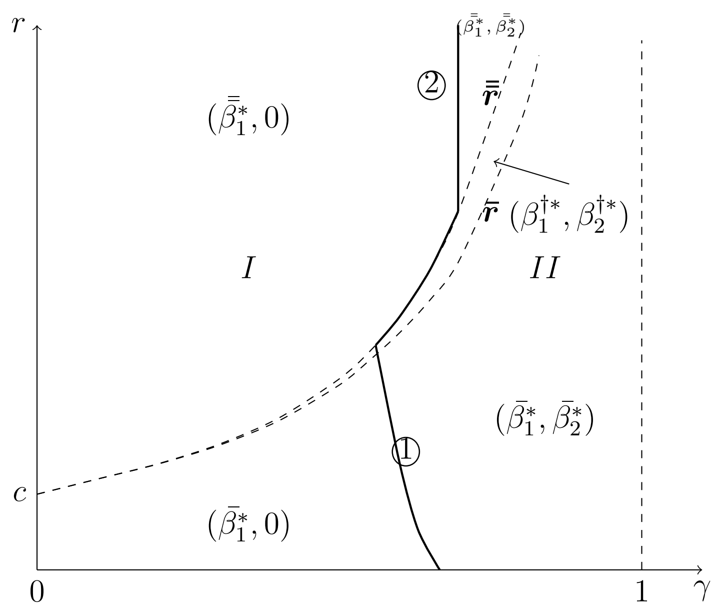

**Figure 5:** Advertising levels and traffic flow

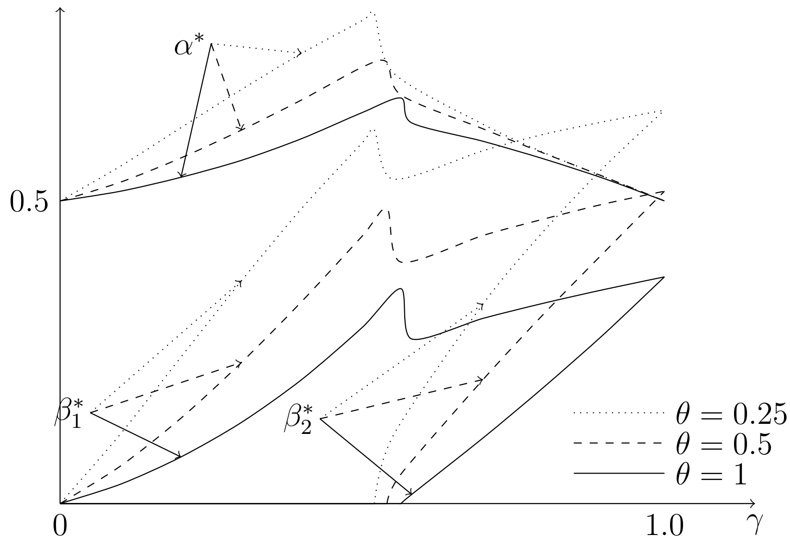


**Figure 6:** Firm profits